Limits of functions: Students solving tasks

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Introduction

This study was conducted to reveal how students at university level justify their solutions to tasks with various degrees of difficulty. The study is part of a larger study of students' concept formation of limits. The mathematical area is limits of functions. The study was carried out at a Swedish university at the first level of mathematics. The results are, however, applicable to other countries as well since students meet similar challenges in their learning of limits. I have, in discussions with some Australian mathematics teachers at university level, found out that the topics taught in basic mathematics courses in Australia are similar to Swedish courses. Two groups of students taking the same course in successive semesters have been solving tasks. Their solutions are categorised here and analysed to create a picture of how students reason about limits.

Background and questions

When students take a course in analysis, they solve a vast number of tasks. They can check if they got correct answers and then continue to work with their problems; but what can be said if some of the solutions are correct for the wrong reasons or wrong by accident despite a thorough attempt? This could lead to serious errors in the students' mental representations of the concept at hand. For a mathematics teacher, to be able to assist his or her students, it is essential to be aware of the different ways students justify their claims. In this study I address the following questions:

- How do students solve problems with limits?
- How do they explain their solutions?

Theoretical background

About problem solving in general

Mathematics is often expressed with symbols operated by certain rules. The rules have to be known to an individual engaged in mathematical activity and they can be memorised. This is not enough if he or she wants to understand mathematics though. Instead of only memorising formulas and procedures, the individual needs to have an exploring attitude to problem solving (Schoenfeld, 1992). Pólya (1945) describes a way to go about it in terms of decomposing and recombining. The problem is at first considered as a whole, then details are examined to give more information for the solution process. The details are combined in different ways and this may give a new perspective to the problem as a whole.

Students learn new and improved methods for problem solving as they take courses in mathematics. This means that they eventually can have quite a few methods to choose from, both new and old ones. When an individual encounters a problem he or she might not fetch the optimum solution method from the mind (Davis & Vinner, 1986; Pólya, 1945). This is not the same as saying that the student cannot solve the problem in a better way. We cannot know what strategies are available in an individual's mind, but we can see the chosen method. The students' actions are shaped by their abilities (Star, 2000). Whatever effective and numerous methods a person has in his or her mind, if they are unreachable at the time they are needed, they are of no use.

A concept image is the total cognitive representation of a notion that an individual has in his or her mind (Tall & Vinner, 1981). It might be partially evoked and different parts can be active in different situations leading to possible inconsistencies. An individual's concept image might differ from the formal concept definition or the concept image in itself can be confusing or incoherent. It may be that an individual is convinced that limits are upper or lower bounds, impossible for the function to pass or even to reach (Cornu, 1991). Then his or her concept image is incompatible with the formal concept definition. Such confusion on a critical issue may provoke the individual to try to understand how it should actually be, but there is also the danger of giving up because of the obstacle.

Lithner (2000; 2003) describes, influenced by Pólya, different types of reasoning in problem solving. He states *plausible reasoning* (PR) to be if the argumentation in the reasoning structure:

(i) is founded on intrinsic mathematical properties of the components involved in the reasoning, and
(ii) is meant to guide towards what probably is the truth, without necessarily having to be complete or correct. (Lithner, 2003, p. 33)

Established experience (EE) is at hand when the argumentation:
(i) is founded on notions and procedures established on the basis of the individual's previous experiences from the learning environment, and

(ii) is meant to guide towards what probably is the truth, without necessarily having to be complete or correct. (Lithner, 2003, p. 34).

Identification of similarities (IS) occurs when the reasoning satisfies the two conditions:

(i) The strategy choice is founded on identifying similar surface properties in an example, theorem, rule, or some other situation described earlier in the text.

(ii) The strategy implementation is carried through by mimicking the procedure from the identified situation. (Lithner, 2003, p. 35).

Lithner's study showed that almost all the time the students in his study spent on mathematics at home was devoted to exercises and IS was the preferred way to reason. The students compared the problems with solutions in the textbook to the problems they should solve and used the strategy from the textbook. One problem for the students was to identify the essential surface properties in order to select the correct procedure for the solution. This way of reasoning has a weak foundation since it is based on memory and not understanding. It is easy to make mistakes. The students did not display much reflection on intrinsic properties or awareness of relations in the study. The author suggests that one reason for this can be that the students are not worried about their inadequate insight into the critical features and therefore do not regard them as useful. This kind of judgement of their own capability has occurred among the students in the study presented in this paper as well (Juter, 2003). Many of the students were unable to solve non-routine tasks or explain given solutions about limits of functions but they thought they had control over the notion of limits anyway. This sense of control can come from successful problem solving (Cornu, 1991) and the difficulties with a few non-routine problems do not change that feeling.

Hiebert and Lefevre (1986) present a different perspective. They speak of conceptual and procedural knowledge. *Conceptual knowledge* has an emphasis on relations. The items of a notion are connected through relations and together they form a mental web. A part of conceptual knowledge cannot be thought of as a disjointed piece of information. Conceptual knowledge develops via construction of relations between items. The items can be other relations or concepts where the connection can be between two (or more) items that are existing already in the mind or between a new and an existing item. When this connection is created, the result often becomes more than its parts jointly (Dreyfus, 1991). Parts with no prior relations become connected and suddenly more things fit together. Hiebert and Lefevre (1986) present two levels at which relationships between items of mathematical knowledge can be created. At the *primary level,* the connections are of the same or lower degree of abstractness than the items that are connected. At the *reflective level,* the relations are at a higher level than the connected items. At this level, the relationships are often based on identification of similar crucial features in seemingly different items. A relation can be at more than one level. *Procedural knowledge* is divided in two parts (Hiebert & Lefevre, 1986). One is knowledge

of the formal language of mathematics, that is the symbols and words we use to represent the mathematics. Knowing this is not the same as knowing what the mathematics described actually means. It is just the accepted forms that are known. The other part consists of the algorithms used for solving tasks. Characteristics of these algorithms are their step-by-step descriptions which are similar to an assembly description for a shelf. There are no relational needs except that one instruction follows the former one.

Mathematical knowledge needs links between conceptual and procedural knowledge and both of them are essential for students to be able to perform mathematics satisfactorily as I see it. Otherwise, the result might be that they are able to produce an answer to a task but not to understand what they have done. On the other hand students can have an accurate intuition for the mathematics, but be unable to make the necessary calculations. Such links can make it easier to remember algorithms and when to use each of them. If there is a connection between an algorithm and the explanation to it, it is a big help for the students when they are solving problems. Links to the underlying meaning of the symbols are also important for an individual to be able to understand what is actually going on. Then it might be possible to rephrase the task in an easier way and to use a preferred algorithm rather than a perhaps harder one tied to the initial formulation of the task. Connections from conceptual to procedural knowledge in the use of symbols make the thinking smoother, according to Hiebert and Lefevre (1986). With a compact symbolic representation it is easier to manipulate objects and processes and detect their features. In this way, new knowledge can be formed. Powerful procedures create more space for other thoughts necessary for completion of the task.

Mathematical thinking in a problem solving situation is a dynamic process and the relations between items of conceptual and procedural knowledge are therefore vital (Silver, 1986). It is very rare that a person only shows evidence of one sort of knowledge. Problem solving is an action involving different kinds of knowledge. Non-standard tasks, but also easier tasks, can reveal relations between the different knowledge types. The former types generally require understanding or interpretation of some kind and thereby the interplay becomes more visible.

School mathematics has a tradition of focusing on manipulation of symbols rather than on understanding what the symbols represent (Davis, 1986). Davis suggests that experience of the mathematical area should come before the symbols for it are introduced. Then the meaning of the symbols will be familiar from the start and this will perhaps make problem solving less demanding.

I consider the students' reasoning and problem solving to be expressions of their concept images (Tall & Vinner, 1981).

About problems related to limits

Infinity is a notion that can cause trouble. It is something for which an individual has one or several intuitive representations (Tall, 1980). If there are multiple, different representations evoked simultaneously, the result might

be erroneous. When dealing with limits of functions one has no specific method or algorithm as one has for diofantic equations for example. The limit process appears potentially infinite and students can get the impression that there is no end to it (Tall, 2001). It can be hard for them to work with items that are confusing in identity. Is it an object or a process?

A common error in students' concept interpretations of limits of functions is that functions do not attain their limit values (Cornu, 1991; Tall, 1993; Tall, 2001; Szydlik, 2000). There is also a possible mix-up of $f(a)$ and

$$\lim_{x \to a} f(x)$$

(Davis & Vinner, 1986). These two flaws combined can totally block students in their struggle with tasks that could easily be solved with an equation for instance.

When students meet the concept of limits at universities for the first time they have already been working with functions and their graphs. The goals in the curriculum for upper secondary school in Sweden do not mention limits explicitly, but the students are expected to learn about derivatives and integrals (Skolverket, 2003). This implies that limits of functions are discussed in some form. The students at universities therefore have an existing concept image of limits of functions that has been satisfactory in the contexts they have worked in so far. Hence there is no need to learn the formal limit concept to be able to analyse functions (Williams, 1991). The students have to experience the need for further sophistication in their mathematical development to adjust their perhaps blunt existing mental representation of limits. Szydlik's (2000) data implies that there is a connection between the understanding of limits and functions and infinitesimals. Students who used infinitesimal language in their descriptions of limits did not show evidence of increasing alternate conceptions of limits. Their ability to solve problems about limits was not diminished either. Infinitesimals can on the other hand have the opposite effect too. Milani and Baldino (2002) found cases where students had trouble with their concept images of infinitesimals and limits. Images and definitions of limits were different but this was not the case for infinitesimals. The authors ask the question whether students perceive definitions as useful for their mathematical activities.

The definition of the limit concept often causes difficulties for students (Cornu, 1991; Juter, 2003; Vinner, 1991). The students' concept definitions are not always compatible with the formal concept definition and that can cause an incoherent concept image with different rules for different situations. If a problem is stated in a manner that is not specifically represented in the students' concept images, there can be more than one representation evoked in the effort to solve the problem. This can make the students confused and unable to proceed.

There are many things that can disturb the solving process. The goal of this study is to find out more about the students' solution strategies when solving problems involving limits of functions and the justifications of their choices.

The study and the students

The students in this study were enrolled in a 20 week, full time course in mathematics. They were learning analysis and algebra at basic University level. In the Spring Semester of 2002, 111 students solved Task 1 to Task 3 described below. They had treated limits of functions in the course and it was nothing new for them. Eleven days later they were given Task 4 and Task 5. There were 87 students who participated in that session. The last set of tasks provided solutions that could be wrong or incomplete. This was stated on the sheet with the tasks and the students were to give a complete and correct solution to each task. A new group of 78 students tackled the same tasks the following Semester (Autumn of 2002). They were all given the five tasks at the same time, after the notion of limits of functions had been dealt with. The group was smaller than the previous Semester and one reason is that the students who had biology or chemistry as a main topic were offered another course more suitable for them but this was not an option for the students in the Spring Study. In both cases, the tasks formed part of three questionnaires along with questions about limits and attitudes towards mathematics in general. Two interviews with each of 15 of the students were conducted in the Autumn Study (Juter, 2003). The first three tasks in this presentation were slightly altered in the Second Study since many of the students misinterpreted or did not understand what the tasks were about. The change was from "Can the function $f(x) = 2x + 3$ attain the limit value?" to "Can the function $f(x) = 2x + 3$ attain the limit value in 1a?" with respective functions in Task 1 to Task 3 below.

Method

The tasks were constructed to focus on different aspects of the limit concept. The degree of difficulty varied to identify the level the students could handle. I explained what I wanted the students to do and at each session they responded to the questionnaires to make sure that it was clear to them.

The collected data has been rewritten and categorised with the aid of the computer program NUD*IST (N6, 2003). The categories were decided from the raw material. I did not create them in advance, other than that the right and wrong answers made different categories. There were subcategories in each of them based, on the students' justifications of their responses. They were chosen from the different reasons the students used for their solutions. This process led to a number of categories. Categories with similar types of reasoning were merged together to make the presentation more accessible. Some solutions are in more than one category since some students gave more than one solution or solutions that fitted more than one category for other reasons. This way to work with the data gave different types of category systems for the different tasks.

Empirical data

Examples of typical student answers are provided in a table after each task. The tables also include the number of students from each semester in each category. The numbers within brackets are percentages of participating students in each class. (R) indicates that the answer is right and (W) denotes a wrong answer.

Task 1

a) Decide the limit: $\lim_{x \to 3}(2x+3)$.

b) Explanation.

c) Can the function $f(x) = 2x + 3$ attain the limit value in 1a?

d) Why?

Table 1(i). Typical student answers in the categories for Task 1a–b. Number of students (%).

Category	1a	1b	Spring 2002	Autumn 2002
Tends to (R)	9	x tends to 3 $2x$+3 becomes close to 9	39 (35)	33 (42)
Replace x by 3 (R)	9	$2 \times 3 + 3 = 9$	37 (33)	19 (24)
Mixed (R)	9	x tends to 3 and $2 \times 3 + 3 = 9$	9 (8.1)	9 (12)
Theory (R)	9	Continuous function that attains the function value in the point 3	7 (6.3)	3 (3.8)
No explanation (R)	9	–	9 (8.1)	8 (10)
Wrong or empty		The answer is either wrong or missing	12 (11)	16 (21)

Table 1(ii). Typical student answers in the categories for Task 1c–d. Number of students (%).

Category	1c	1d	Spring 2002	Autumn 2002
Theory (R)	Yes	The function is continuous in the point	22 (20)	23 (29)
Replace x by 3 (R)	Yes	$2x + 3 = 9$ for $x = 3$	22 (20)	21 (27)
No explanation (R)	Yes		15 (14)	8 (10)
Limits not attainable (W)	No	A function does not attain the limit value, it only comes very close, it is in the definition	9 (8.1)	10 (13)
No reason (W)	No	–	3 (2.7)	3 (3.8)
Empty or misinterpretation		The answer has no connection to the question or is missing	40 (36)	16 (21)

Task 2

a) Decide the limit: $\lim\limits_{x\to\infty}\dfrac{x^3-2}{x^3+1}$.

b) Explanation.

c) Can the function $f(x)=\dfrac{x^3-2}{x^3+1}$ attain the limit value in 2a?

d) Why?

Table 2 (i). Typical student answers in the categories for Task 2a–b. Number of students (%).

Category	**2a**	**2b**	**Spring 2002**	**Autumn 2002**
Exclude –2 & 1 (R)	1	The x^3 terms dominate, –2 and 1 insignificant as $x \to \infty$	53 (48)	32 (41)
Algebra (R)	1	Divide with x^3, $\dfrac{\frac{x^3}{x^3}-\frac{2}{x^3}}{\frac{x^3}{x^3}+\frac{1}{x^3}}=\dfrac{1-\frac{2}{x^3}}{1+\frac{1}{x^3}}\to 1$	17 (15)	28 (36)
No explanation (R)	1	–	6 (5.4)	4 (5.1)
Algebra (W)	–2	x^3 cancel out since it is the same number	17 (15)	12 (15)
Infinity reason (W)	Does not exist	Divided by infinity	11 (10)	0 (0)
Empty or misinterpretation		The answer has no connection to the question or is missing	10 (9)	9 (12)

Table 2 (ii). Typical student answers in the categories for Task 2c–d. Number of students (%).

Category	**2c**	**2d**	**Spring 2002**	**Autumn 2002**
$x^3 - 2 \neq x^3 + 1$ (R)	No	$x^3 - 2$ can never be equal to $x^3 + 1$ for the same value of x	7 (6.3)	17 (22)
–2 & 1 (R)	No	terms –2 and 1 will always remain	7 (6.3)	5 (6.4)
No explanation (R)	No	–	18 (16)	8 (10)
Infinity reason (W)	No	Since x never attains the value ∞	16 (14)	21 (27)
Theory (W)	No	The function tends to the limit value, it does not attain it	7 (6.3)	6 (7.7)
No reason (W)	Yes	–	15 (14)	7 (9.0)
Empty or misinterpretation		The answer has no connection to the question or is missing	42 (38)	17 (22)

Task 3

a) Decide the limit: $\lim_{x \to \infty} \frac{x^5}{2^x}$.

b) Explanation.

c) Can the function $f(x) = \frac{x^5}{2^x}$ attain the limit value in 3a?

d) Why?

Table 3 (i). Typical student answers in the categories for Task 3a–b. Number of students (%).

Category	**3a**	**3b**	**Spring 2002**	**Autumn 2002**
Exp. dominant (R)	0	2^x grows faster than x^5	80 (72)	64 (82)
No explanation (R)	0	–	6 (5.4)	3 (3.8)
Infinity or no limit (W)	∞	x^5 is larger than 2^x when x is large	11 (10)	7 (9.0)
Empty or misinterpretation		The answer has no connection to the question or is missing	17 (15)	4 (5.1)

Table 3 (ii). Typical student answers in the categories for Task 3c–d. Number of students (%).

Category	**3c**	**3d**	**Spring 2002**	**Autumn 2002**
$x = 0$ (R)	Yes	For $x = 0 \rightarrow f(0) = 0/1 = 0$	15 (14)	16 (21)
No explanation (R)	Yes	–	5 (5)	6 (7.7)
Does not reach limit (W)	No	We can only get infinitely close	14 (13)	16 (21)
$x^5 \neq 0$ or $0/0$ (W)	No	Then the numerator has to be zero and it never is	22 (20)	12 (15)
Right for wrong reason (W)	Yes	Because the denominator attains a much larger number for large x	5 (5)	16 (21)
Empty or misinterpretation		The answer has no connection to the question or is missing	47 (42)	12 (15)

Task 4:

Problem: Decide the following limit value: $\lim_{x\to 1}\frac{x^2+x}{x^2-1}$.

The students were given the following:

Solution: $\frac{x^2+x}{x^2-1}=\frac{x(x+1)}{(x-1)\,(x+1)}=\frac{x}{x-1}\to\infty$ when x → 1.

The task was for the students to decide the proper adjustments to make the solution correct:

Adjustments (What changes or complements are needed and why?):

Table 4. Typical student answers in the categories for Task 4. Number of students (%).

Category	**4**	**Spring 2002**	**Autumn 2002**
Both sides (R)	$x\to 1$ from minus or plus	18 (21)	28 (36)
One side (R) Incomplete	As $x\to 1$ the denominator becomes negative	10 (11)	5 (6.4)
Dominant factor (W)	$\frac{x^2\left(1+\frac{1}{x}\right)}{x^2\left(1-\frac{1}{x^2}\right)}\to\frac{1+1}{1-1}\to\infty$	15 (17)	12 (15)
Reasoning (W)	x tends to 1 hence it can not be infinite	14 (16)	18 (23)
No change (W)	It is entirely correct!	8 (9.2)	4 (5.1)
Empty or unclear	The answer is missing or does not make any sense	24 (28)	17 (22)

Task 5

Problem: Decide the following limit value: $\lim_{x\to\infty} x\sin\left(\frac{2}{x}\right)$.

The students were given the following:

Solution: $x\sin\left(\frac{2}{x}\right)=\frac{\sin\left(\frac{2}{x}\right)}{\frac{1}{x}}$.

We know that $\frac{2}{x}\to 0$ when $x\to\infty$ and $\frac{1}{x}\to 0$ when $x\to\infty$.

The limit value $\lim_{x\to 0}\frac{\sin x}{x}=1$ implies that $\frac{\sin\left(\frac{2}{x}\right)}{\frac{1}{x}}\to 1$ when $x\to 0$.

The task was for the students to decide the proper adjustments to make the solution correct:

Adjustments (What changes or complements are needed and why):

Table 5. Typical student answers in the categories for Task 5. Number of students (%).

Category	**5**	**Spring 2002**	**Autumn 2002**
All right (R)	$\frac{\sin\left(\frac{2}{x}\right)}{\frac{1}{x}} = \frac{\sin\left(\frac{2}{x}\right)}{\frac{2}{x}} \cdot 2 \to 2$	10 (11)	17 (22)
Part right (R) Incomplete	$\frac{2}{x}$ is not the same thing as $\frac{1}{x}$ therefore you cannot use limit values on limit values	9 (10)	11 (14)
Reasoning (W)	$\frac{1}{x} \to 0$ when $x \to \infty$ so the denominator has to go $\to 0$ and the expression $\to \infty$	31 (36)	18 (23)
No change (W)	Nothing	4 (4.6)	1 (1.3)
Empty or unclear	The answer is missing or does not make any sense	33 (38)	32 (41)

Analysis

The students' solutions to Task 1a–b are mainly correct. The correct solutions were divided into categories, with most responses falling into two categories. The second category explicitly suggests that the limit value is attainable and a large proportion of the students have chosen this way to solve the task. An even larger number of the students solved the task in words of approaching the limit value. The idea that limits are not attainable comes again in Task 1c–d where students use this as an argument for the function not to attain the value 9. It is a documented fact that some students have this misinterpretation of the limit value definition (Cornu, 1991; Tall, 1991). Some students do not separate the part with the limit value from the part with the function, that is they mix up *f(a)* and $\lim_{x \to a} f(x)$ as Davis and Vinner (1986) describe. If the students have the conviction that limits are unattainable, there might be problems analysing the function. This is something that follows through Task 1 to Task 3. A large part of the students did not answer or answered Task 1c–d in a way that did not make any sense. Some wrote that it depends on what x tends to and this is the reason for the additional words in the formulation of the task in the autumn of 2002. The result of the change is that this category

contained a smaller number of students the second semester. This applies for all the c–d questions.

Task 2 is harder for the students to handle and one serious problem is algebra. There are several students who think that x^3 in the numerator and the denominator are cancelling out in a way that erases the terms and leaves only the constants or that x^3 can be replaced by 1 with a similar reasoning. There are more students in the autumn study using algebra in a correct manner to solve the task than in the spring study. The majority of the students are unable to solve Task 2c–d. One problem is attainability as discussed above. Only some students regard this problem as an equation to solve and this is obvious in Task 3c–d as well. Twenty-two percent of the students in the autumn study solved Task 2c–d with an equation in a correct way whereas the corresponding figure in the Spring Study is 6.3. Eight students from each semester abused the equal sign and wrote

$$\frac{x^3-2}{x^3+1}=\frac{x^3}{x^3}$$

in their explanations in Task 2b. Their results are in the column "Exclude –2 & 1 (R)".

Task 3a–b is a standard limit value that is well known to the students apart from about 10% who got it backwards ("Infinity or no limit (W)"). There are fewer categories for this task since over 70% of all the students used the standard limit value reasoning. The problems come in Task 3c–d where the mix up of limits and functions is clear. Students claim that x is never zero since x tends to infinity, but this has nothing to do with the functions ability to attain the value 0. Algebra is a problem for some students here too. Some believe that $2^0 = 0$, for example.

Task 4 offers a challenge for many students. Forty-six of the 165 students were able to solve the task completely. One mistake many made was to use the dominant factor and divide, but that is not solving the problem with left and right limit value. This method is often used when x tends to infinity and the students seem to just go through the motions without considering the characteristics of the task they are involved with.

Task 5 is apparently the most demanding one since only 26 of all the students managed to solve it properly. The 20 students in the "Part right (R) Incomplete" category were also correct but they did not give the limit value, they only pointed at the inaccuracy so they might or might not be able to carry out the calculations. The majority of the students either left the task unsolved or reasoned incorrectly.

Discussion

The tables show that the students' explanations of choices of solutions vary. They are good at finding limits in the first three tasks. The problems seem to come when the tasks are a bit different from what the students are used to. A task can be a problem for one student but just routine for another

(Björkqvist, 2001; Grevholm, 1991) and this obviously leads to different outcomes for the students. Part c and d are not mathematically more demanding than the other parts of Task 1 to Task 3 but there is something that troubles the students in them. The outcome might have been different if the parts were not presented together. The students were working in the context of limits when they were asked to examine the functions for attainability. The effect was that the functions were only considered locally in some cases and the misconception that limits are unattainable (Cornu, 1991; Tall, 1993; Szydlik, 2000) made some students claim that the function could not attain the value even if it obviously could (Task 1 and Task 3). There were also other types of confusion. One student from the Study in the Autumn of 2002 answered Task 1 like this:

1a: $\to 9$
1b: If $x \to$ (=3) the result becomes 9, the function will never attain that.
1c: Yes.
1d: If you let $x \to 3$.

The confusion of functions with limits of functions is a problem that indicates a lack of relations between the concepts. If the students were more confident about the roles and possibilities of the notions they would have a better chance to solve problems correctly. An insufficient mathematical base to work from can cause constraints on the individual, in that he or she is not sure what operations are allowed and how to carry them out. This uncertainty can be the reason for the many empty answers.

Infinity is obviously an element that can cause confusion (Tall, 1980). All tasks revealed problems with infinity in different degrees. One thing that is connected with infinity is the notion of local limits in a wider context. Table 5 indicates this by the categories "Reasoning (W)" and "No change (W)". Many students are reasoning about the local limits for the functions

$$\frac{2}{x} \text{ and } \frac{1}{x}$$

separately or dissect the given function in other ways and locally consider limits. The students follow part of Pólya's (1945) model with decomposing and recombining, but the recombining to check at the whole again is overlooked. The students' reasoning hints an attempt for plausible reasoning or established experience (Lithner, 2003), but there appears to be a lack of parts in the mental web that represents this fraction of the concept image (Tall & Vinner, 1981) since they do not have access to the essential information about the properties of the limit process and functions. The development of conceptual knowledge (Hiebert & Lefevre, 1986) has not had a satisfactorily progress.

Table 4 shows an example of identification of similarities (Lithner, 2003) as the students in the category "Dominant factor (W)" use a solving technique that is usually effective on rational functions as x tends to infinity. Here the students recognise the rational function but they do not consider what x tends to. The resemblances at first sight are not the same on all crucial points

and the chosen method is not working any better than the given solution. This task requires reasoning rather than an algorithm.

The choices of methods seem to be triggered by first sight resemblances in other cases too. There are comparisons with standard limit values. Sometimes the method is working, as it did for most students' solutions to Task 3a–b. Other times it does not work, as for some suggested solutions to Task 5. The students do not appear to have a global view of the important characteristics of the mathematics at hand. The effect of this can be that critical features are overlooked and the solution is beyond the possibility to reach for the students.

Table 2(ii) and Table 3(ii) show examples of solutions with correct answer and wrong explanation. The students from the autumn study do this to a higher extent than the other students which can be due to various reasons since the two groups learn mathematics under different circumstances. This is something that students must be confronted with to be able to repair. The textbook only gives the answer and not a full solution to the tasks so the first confrontation is in the worst scenario at the exam. If the students have used the wrong arguments for a long time, an adjustment can be hard to make. The students represented in Table 2(ii) who have answered correctly with no explanation can also belong to the category of students with correct answer for the wrong reasons since we do not know why they answered the way they did.

Algebra is the reason for a number of mistakes. Many of the algebraic errors are serious. Table 2 (i) shows the existence of some of them, but there are similar errors in other places too. Some examples are:

$$\frac{2}{0}=2$$

$$\frac{x^3-2}{x^3+1}=\frac{\frac{x^3}{x^3}-2}{\frac{x^3}{x^3}+1}=\frac{1-2}{1+1}=-2$$

$$\frac{x^3-2}{x^3+1}=\frac{-2}{1}=-2$$

$$\frac{x^3-2}{x^3+1}=\frac{x^3}{x^3}\cdot\frac{1-2}{1+1}=-\frac{1}{2}$$

These examples expose a lack of knowledge in basic calculation rules that should not exist at university level. Investigations at upper secondary school and among student teachers in mathematics show that students find it hard to work with algebra and similar mistakes as those in this study are common (Grevholm, 2003; Olteanu, Grevholm & Ottosson, 2003). If the solution process is hindered by such matters, there is much for the students to work with to improve their concept images. The students have to be aware of the problems before they feel a need to alter anything and if the errors are not discovered nothing will happen.

Conclusion

I have shown students' different types of solutions to tasks about limits of functions and how they are explained. A variety of solutions of different accuracy appeared in the study. I did not expect to see so much confusion about functions and limits of functions or the problems caused by algebra. These are examples of problems related to what is allowed and what is not. The large number of correct solutions based on wrong facts is also a serious problem that shows that, for many students, connections between concepts are wrong or not there at all. Perhaps students need to experience a larger variety of problems to understand the rules and properties of mathematics better. Problems that demand thought and provoke the students' concept images give opportunities for the students to make appropriate adjustments. Then they would probably get to know vital intrinsic properties for notions at a deeper level as well. Large groups of students and time shortage make this kind of extra effort to help students very hard to carry out, but students should be able to see the core of the mathematics they work with and to recognise its characteristic features. These abilities can be developed if teachers can provide an environment that inspires the students to discuss mathematical issues.

Despite all errors and misunderstandings that have been documented here, there are skilful problem solvers among the students and that is something we must not forget. One goal of our teaching is to make able problem solvers of as many students as possible. We need not only to know how the mathematically weak students reason but also how the mathematically developed students do it.

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